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Bending solutions of Levinson beams and plates in terms of the classical theories

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The paper is dedicated to Prof. Robert M. Jones on the occasion of his 60th birthday celebration

Abstract

Using the mathematical similarity of the governing equations of the classical beam and plate theories and the Levinson beam and plate theories, and the basis of load equivalence, exact relationships between the bending solutions of the two theories for beams and plates are derived. These relationships enable the conversion of the well-known classical (Euler–Bernoulli) beam and (Kirchhoff) plate solutions to their shear deformable counterparts using the Levinson beam and plate theories. Examples are given to illustrate the use of these relationships. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Two-dimensional plate theories can be classified into two types: (1) classical (Kirchhoff) plate theory (CPT), in which the transverse shear deformation effects are neglected and (2) shear deformation plate theories. The one-dimensional counterpart of the Kirchhoff plate theory is the Euler–Bernoulli beam theory (EBT).

There are a number of shear deformation plate theories. The simplest is the first-order shear deformation plate theory (FSDT), also known as the Mindlin plate theory (Mindlin, 1951). The FSDT extends the kinematics of the CPT by including a gross transverse shear deformation in its kinematic assumptions, i.e. the transverse shear strain is assumed to be constant with respect to the thickness coordinate. In the FSDT, shear correction factors are introduced to correct for the discrepancy between the actual transverse shear force distributions and those computed using the kinematic relations of the FSDT. The one-dimensional counterpart of the Mindlin plate theory is known as the Timoshenko beam theory (TBT).

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Second- and higher-order shear deformation plate theories use higher-order polynomials in the expansion of the displacement components through the thickness of the plate. The higher-order theories introduce additional unknowns that are often difficult to interpret in physical terms. There are a number of third-order plate theories in the literature, and a review of these theories is given in the text book by Reddy (1984b).

A third-order plate theory is based on the following displacement field or its equivalent:

$$u(x,y,z) = z\phi_x(x,y) - \alpha z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \tag{1.1a}$$

$$v(x,y,z) = z\phi_y(x,y) - \alpha z^3\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \tag{1.1b}$$

$$w(x,y,z) = w_0(x,y), \tag{1.1c}$$

where $\alpha = 4/(3h^2)$. Note that by setting $\alpha = 0$ in Eqs. (1.1a)–(1.1c), we recover the displacement field of the FSDT. The displacement field accommodates a quadratic variation of the transverse shear strains (and hence shear stresses) through the thickness and the vanishing of transverse shear stresses on the top and bottom surfaces of the plate. Unlike the FSDT, a third-order plate theory requires no shear correction factors. Fig. 1 shows the positive coordinate directions, stress resultants, and kinematics of deformation of an edge in various theories.

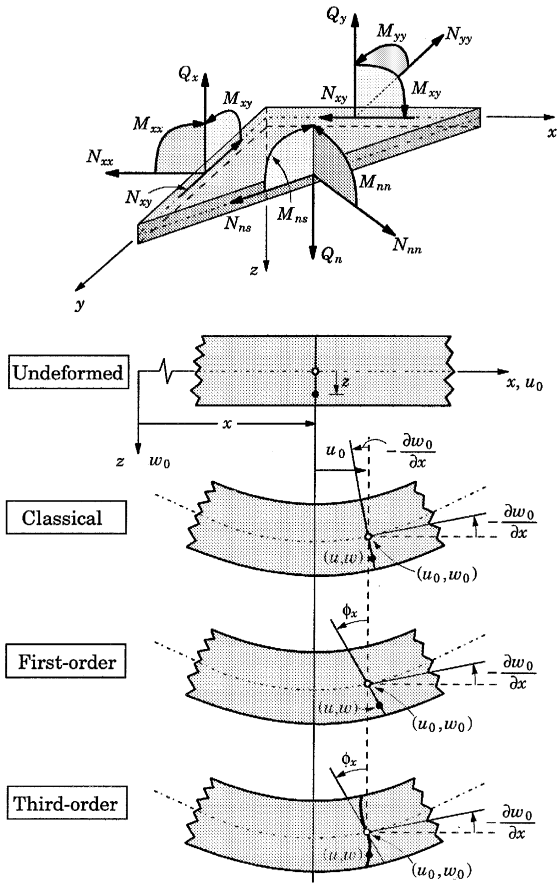

Fig. 1. The positive coordinate directions, stress resultants, and kinematics of deformation of an edge in various theories.

Levinson (1981, 1980) used a vector approach to derive the equations of equilibrium of isotropic beams and plates. Reddy (1984a) independently developed a third-order laminate plate theory and derived equations of motion associated with the displacement field (1.1a)–(1.1c) using the principle of virtual displacements. Bickford (1982) and Heyliger and Reddy (1988) also derived a variationally consistent third-order beam theory, much in the same way Reddy did for plates. The Levinson plate theory (LPT) is a lower-order theory [1] than the Reddy plate theory, and contains no higher-order stress resultants. For other pertinent works on theories of plates, the reader may consult the text books of Reddy (1984b, 1997, 1999) and references therein (also see Wang and Kitipornchai, 1999).

In this study, bending relationships between the Levinson beam theory (LBT) and the EBT and the LPT (a simplified version of the Reddy third-order plate theory) and the Kirchhoff plate theory are developed. Numerical examples are presented to show how the solutions of the Levinson plate equations are obtained directly from the Kirchhoff plate solutions for rectangular plates with two parallel edges simply supported while the other two edges may have arbitrary boundary conditions (Lévy solutions).

2. Beams

2.1. Governing equations

The equilibrium equation of the EBT is

$$-\frac{\mathrm{d}^2 M_{xx}^{\mathrm{E}}}{\mathrm{d}x^2} = q \quad \text{for } 0 < x < L. \tag{2.1}$$

It is useful to introduce the shear force Q_x^{E}

$$Q_x^{\mathrm{E}} = \frac{\mathrm{d}M_{xx}^{\mathrm{E}}}{\mathrm{d}x}, \tag{2.2}$$

and rewrite the equilibrium equation (2.1) in the form

$$-\frac{\mathrm{d}Q_x^{\mathrm{E}}}{\mathrm{d}x} = q, \tag{2.3}$$

where superscript E refers to quantities belonging to the EBT.

The form of the boundary conditions of the Euler–Bernoulli theory is either the displacement w_0^{E} is known or the shear force $Q_x^{\mathrm{E}} = \mathrm{d}M_{xx}^{\mathrm{E}}/\mathrm{d}x$ is specified at a point on the boundary. In addition, either the slope $\mathrm{d}w_0^{\mathrm{E}}/\mathrm{d}x$ is specified or the bending moment M_{xx}^{E} is known at a boundary point. Thus, we have

$$\text{Specify}: \quad \left\{ \begin{matrix} w_0^{\mathrm{E}} \\ \frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x} \end{matrix} \right\} \quad \text{or} \quad \left\{ \begin{matrix} Q_x^{\mathrm{E}} = \frac{\mathrm{d}M_{xx}^{\mathrm{E}}}{\mathrm{d}x} \\ M_{xx}^{\mathrm{E}} \end{matrix} \right\}. \tag{2.4}$$

The strain–displacement relations of the Levinson–Bickford–Reddy beam theories are given by

[1] The order referred to here is the total order of differential equations of the theory. The Levinson theory is a sixth-order theory, whereas the Reddy theory is an eighth-order theory.

$$\varepsilon_{xx} = z\frac{\mathrm{d}\phi_x^{\mathrm{L}}}{\mathrm{d}x} - \alpha z^3\left(\frac{\mathrm{d}\phi_x^{\mathrm{L}}}{\mathrm{d}x} + \frac{\mathrm{d}^2 w_0^{\mathrm{L}}}{\mathrm{d}x^2}\right), \tag{2.5a}$$

$$\gamma_{xz} = \left(1 - \beta z^2\right)\left(\phi_x^{\mathrm{L}} + \frac{\mathrm{d}w_0^{\mathrm{L}}}{\mathrm{d}x}\right), \tag{2.5b}$$

where superscript L refers to quantities belonging to the LBT, and

$$\alpha = \frac{4}{3h^2}, \qquad \beta = 3\alpha = \frac{4}{h^2}. \tag{2.6}$$

Instead of using the variationally derived equations of equilibrium, here we use the thickness-integrated equations of elasticity. These are exactly the same as those of the TBT:

$$-\frac{\mathrm{d}M_{xx}^{\mathrm{L}}}{\mathrm{d}x} + Q_x^{\mathrm{L}} = 0, \qquad -\frac{\mathrm{d}Q_x^{\mathrm{L}}}{\mathrm{d}x} = q. \tag{2.7}$$

The stress resultant–displacement relations for the LBT are given by

$$M_{xx}^{\mathrm{L}} = \bar{D}_{xx}\frac{\mathrm{d}\phi_x^{\mathrm{L}}}{\mathrm{d}x} - \alpha F_{xx}\frac{\mathrm{d}^2 w_0^{\mathrm{L}}}{\mathrm{d}x^2}, \tag{2.8}$$

$$Q_x^{\mathrm{L}} = S_{xz}\left(\phi_x^{\mathrm{L}} + \frac{\mathrm{d}w_0^{\mathrm{L}}}{\mathrm{d}x}\right), \tag{2.9}$$

where

$$\bar{D}_{xx} = D_{xx} - \alpha F_{xx}, \tag{2.10a}$$

$$(D_{xx}, F_{xx}) = \int_A (z^2, z^4) E_x \,\mathrm{d}A, \tag{2.10b}$$

$$S_{xz} = \int_A \left(1 - \beta z^2\right) G_{xz}\,\mathrm{d}A. \tag{2.10c}$$

The form of the boundary conditions for the LBT are taken to be the same as those for the TBT, and they are

$$\text{Specify}: \quad \begin{Bmatrix} w_0^{\mathrm{L}} \\ \phi_x^{\mathrm{L}} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} Q_x^{\mathrm{L}} \\ M_{xx}^{\mathrm{L}} \end{Bmatrix}. \tag{2.11}$$

2.2. Relationships

Here, we develop the relationships between the bending solutions of the EBT and the LBT. At the outset, we note that both the EBT and the TBT are fourth-order theories, whereas the variationally derived Bickford–Reddy beam theory (BRT) (Reddy, 1999; Wang et al., 2000) is a sixth-order theory. However, the LBT is a fourth-order theory. Therefore, the relationships between LBT and EBT are similar, as will be shown shortly, to those between TBT and EBT (see Wang, 1995).

First, we note that Eqs. (2.2) and (2.7) together yield

$$Q_x^{\mathrm{L}} = Q_x^{\mathrm{E}} + C_1, \tag{2.12}$$

$$M_{xx}^{\mathrm{L}} = M_{xx}^{\mathrm{E}} + C_1 x + C_2. \tag{2.13}$$

From the stress resultant–displacement relationships in Eqs. (2.2), (2.3), (2.8) and (2.9), we obtain

$$\bar{D}_{xx}\frac{d\phi_x^L}{dx} - \alpha F_{xx}\frac{d^2w_0^L}{dx^2} = -D_{xx}\frac{d^2w_0^E}{dx^2} + C_1x + C_2, \tag{2.14}$$

$$S_{xz}\left(\phi_x^L + \frac{dw_0^L}{dx}\right) = -D_{xx}\frac{d^3w_0^E}{dx^3} + C_1, \tag{2.15}$$

where the stiffness coefficients D_{xx}, F_{xx}, and S_{xz} are defined in Eqs. (2.10b) and (2.10c), and α and β are defined in Eq. (2.5b). Integrating Eq. (2.14) with respect to x, we arrive at

$$\bar{D}_{xx}\phi_x^L - \alpha F_{xx}\frac{dw_0^L}{dx} = -D_{xx}\frac{dw_0^E}{dx} + C_1\frac{x^2}{2} + C_2x + C_3. \tag{2.16}$$

Solving Eq. (2.16) for dw_0^L/dx, substituting it into Eq. (2.15), and solving for ϕ_x^L, we obtain

$$\phi_x^L = -\frac{dw_0^E}{dx} + \frac{\alpha F_{xx}}{D_{xx}S_{xz}}\left(Q_x^E + C_1\right) + \frac{1}{D_{xx}}\left(C_1\frac{x^2}{2} + C_2x + C_3\right). \tag{2.17}$$

Returning to Eq. (2.15) and solving for dw_0^L/dx, we obtain

$$\frac{dw_0^L}{dx} = \frac{dw_0^E}{dx} + \frac{\bar{D}_{xx}}{S_{xz}D_{xx}}\left(Q_x^E + C_1\right) - \frac{1}{D_{xx}}\left(C_1\frac{x^2}{2} + C_2x + C_3\right), \tag{2.18}$$

and on integration, we arrive at the deflection relationship

$$w_0^L = w_0^E + \frac{\bar{D}_{xx}}{S_{xz}D_{xx}}\left(M_x^E + C_1x\right) - \frac{1}{D_{xx}}\left(C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4\right). \tag{2.19}$$

This completes the derivation of the relationships between the solutions of the EBT and the LBT. The constants of integration, C_1, C_2, C_3, C_4 appearing in Eqs. (2.12)–(2.19) are determined using the boundary conditions. For free (F), simply supported (S) and clamped (C) ends, the boundary conditions are given by

$$\mathrm{F}: M_{xx}^E = M_{xx}^L = Q_x^E = Q_x^L = 0, \tag{2.20}$$

$$\mathrm{S}: w_0^E = w_0^L = M_{xx}^E = M_{xx}^L = 0, \tag{2.21}$$

$$\mathrm{C}: w_0^E = w_0^L = \frac{dw_0^E}{dx} = \phi_x^L = 0. \tag{2.22}$$

2.3. Examples

Here, we present two examples to derive the solutions of the LBT using the relationships derived in Section 2.2 and the solutions of the EBT.

2.3.1. Simply supported beam

Consider a simply supported beam of length L and subjected to uniformly distributed load of intensity q_0. The stress resultants and the deflection of the Euler–Bernoulli beam are

$$Q_x^E(x) = \frac{q_0L}{2}\left(1 - 2\frac{x}{L}\right), \tag{2.23}$$

$$M_{xx}^E(x) = \frac{q_0L^2}{2}\frac{x}{L}\left(1 - \frac{x}{L}\right), \tag{2.24}$$

$$w_0^{\mathrm{E}}(x) = \frac{q_0 L^4}{24 D_{xx}} \left(\frac{x}{L} - \frac{2x^3}{L^3} + \frac{x^4}{L^4} \right). \tag{2.25}$$

Using the relationship for simply supported beams (Wang et al., 2000), the corresponding bending solutions for the Timoshenko beam are

$$Q_x^{\mathrm{T}}(x) = Q_x^{\mathrm{E}}(x) = \frac{q_0 L}{2}\left(1 - 2\frac{x}{L}\right), \tag{2.26}$$

$$M_{xx}^{\mathrm{T}}(x) = M_{xx}^{\mathrm{E}}(x) = \frac{q_0 L^2}{2}\frac{x}{L}\left(1 - \frac{x}{L}\right), \tag{2.27}$$

$$\phi^{\mathrm{T}}(x) = -\frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x} = -\frac{q_0 L^3}{24 D_{xx}}\left(1 - 6\frac{x^2}{L^2} + 4\frac{x^3}{L^3}\right), \tag{2.28}$$

$$w_0^{\mathrm{T}}(x) = w_0^{\mathrm{E}}(x) + \frac{1}{K_{\mathrm{s}} A_{xz}} M_{xx}^{\mathrm{E}}(x) = \frac{q_0 L^4}{24 D_{xx}}\left[\left(\frac{x}{L} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right) + 12\Omega\left(\frac{x}{L} - \frac{x^2}{L^2}\right)\right], \tag{2.29}$$

where superscript T refers to quantities in the TBT, and

$$\Omega = \frac{D_{xx}}{K_{\mathrm{s}} A_{xz} L^2} \tag{2.30}$$

and K_{s} is the shear correction factor introduced in the TBT.

In the case of the LBT the boundary conditions yield $C_1 = C_2 = C_3 = C_4 = 0$. Hence, the solutions are

$$Q_x^{\mathrm{L}}(x) = Q_x^{\mathrm{E}}(x) = \frac{q_0 L}{2}\left(1 - 2\frac{x}{L}\right), \tag{2.31}$$

$$M_{xx}^{\mathrm{L}}(x) = M_{xx}^{\mathrm{E}}(x) = \frac{q_0 L^2}{2}\frac{x}{L}\left(1 - \frac{x}{L}\right), \tag{2.32}$$

$$\phi_x^{\mathrm{L}}(x) = -\frac{dw_0^{\mathrm{E}}}{dx} + \frac{\alpha F_{xx}}{D_{xx} S_{xz}} Q_x^{\mathrm{E}} = -\frac{q_0 L^3}{24 D_{xx}}\left[\left(1 - 6\frac{x^2}{L^2} + 4\frac{x^3}{L^3}\right) - 12\alpha\Lambda\left(1 - 2\frac{x}{L}\right)\right], \tag{2.33}$$

$$w_0^{\mathrm{L}}(x) = w_0^{\mathrm{E}}(x) + \frac{\bar{D}_{xx}}{S_{xz} D_{xx}} M_{xx}^{\mathrm{E}} = \frac{q_0 L^4}{24 D_{xx}}\left[\left(\frac{x}{L} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right) + 12\bar{\Omega}\left(\frac{x}{L} - \frac{x^2}{L^2}\right)\right], \tag{2.34}$$

where

$$\Lambda = \frac{F_{xx}}{S_{xz} L^2}, \qquad \bar{\Omega} = \frac{\bar{D}_{xx}}{S_{xz} L^2}. \tag{2.35}$$

The Bickford–Reddy theory (Reddy and Wang, 1998) has the solution

$$Q_x^{\mathrm{B}}(x) = \left(\frac{q_0 \mu}{\lambda^3}\right)\left[\sinh \lambda x - \tanh\left(\frac{\lambda L}{2}\right)\cosh \lambda x + \frac{\lambda L}{2}\left(1 - 2\frac{x}{L}\right)\right], \tag{2.36}$$

$$M_{xx}^{\mathrm{B}}(x) = M_{xx}^{\mathrm{E}}(x) = \frac{q_0 L^2}{2}\frac{x}{L}\left(1 - \frac{x}{L}\right), \tag{2.37}$$

$$\phi_x^{\mathrm{B}}(x) = -\frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x} + \frac{\alpha F_{xx}}{D_{xx} S_{xz}} Q_x^{\mathrm{B}}, \tag{2.38}$$

$$w_0^{\mathrm{B}}(x) = w_0^{\mathrm{E}}(x) + \left(\frac{q_0 \mu}{\lambda^4}\right)\left(\frac{\bar{D}_{xx}}{S_{xz} D_{xx}}\right)\left[-\tanh\left(\frac{\lambda L}{2}\right)\sinh \lambda x + \cosh \lambda x + \frac{\lambda^2 L^2}{2}\frac{x}{L}\left(1 - \frac{x}{L}\right) - 1\right], \tag{2.39}$$

where superscript B refers to quantities in the BRT, and ($S_{xz} = \bar{A}_{xz}$):

$$\lambda^2 = \frac{\hat{A}_{xz} D_{xx}}{\alpha(F_{xx}\bar{D}_{xx} - \bar{F}_{xx} D_{xx})}, \qquad \mu = \frac{\bar{A}_{xz}\bar{D}_{xz}}{\alpha(F_{xx}\bar{D}_{xx} - \hat{F}_{xx} D_{xx})}, \tag{2.40}$$

$$\begin{aligned} &\bar{D}_{xx} = D_{xx} - \alpha F_{xx}, \qquad \bar{F}_{xx} = F_{xx} - \alpha H_{xx}, \\ &\bar{A}_{xz} = A_{xz} - \beta D_{xz} = S_{xz}, \qquad \bar{D}_{xz} = D_{xz} - \beta F_{xz}, \\ &\hat{A}_{xz} = S_{xz} - \beta \bar{D}_{xz}, \end{aligned} \tag{2.41}$$

$$(A_{xx}, D_{xx}, F_{xx}, H_{xx}) = \int_A (1, z^2, z^4, z^6) E_x \, \mathrm{d}A, \tag{2.42a}$$

$$(A_{xz}, D_{xz}, F_{xz}) = \int_A (1, z^2, z^4) G_{xz} \, \mathrm{d}A. \tag{2.42b}$$

For a rectangular cross-section beam, it can be shown that

$$\frac{\bar{D}_{xx}\bar{D}_{xx}}{\bar{A}_{xz} D_{xx} D_{xx}} = \frac{6}{5A_{xz}}, \qquad \frac{\bar{D}_{xx}}{\bar{A}_{xz} D_{xx}} = \frac{6}{5A_{xz}}. \tag{2.43}$$

2.3.2. Cantilever beam

For a cantilever beam under uniformly distributed load of intensity q_0, the stress resultants and the deflection of the Euler–Bernoulli beam are found to be

$$Q_x^{\mathrm{E}}(x) = q_0 L\left(1 - \frac{x}{L}\right), \tag{2.44}$$

$$M_{xx}^{\mathrm{E}}(x) = -\frac{q_0 L^2}{2}\left(1 - \frac{x}{L}\right)^2, \tag{2.45}$$

$$w_0^{\mathrm{E}}(x) = \frac{q_0 L^4}{24 D_{xx}}\left(6\frac{x^2}{L^2} - 4\frac{x^3}{L^3} + \frac{x^4}{L^4}\right). \tag{2.46}$$

Using the relationships (2.26)–(2.29) for clamped–free (CF) beams, the corresponding bending solutions for the TBT are

$$Q_x^{\mathrm{T}}(x) = Q_x^{\mathrm{E}}(x) = q_0 L\left(1 - \frac{x}{L}\right), \tag{2.47}$$

$$M_{xx}^{\mathrm{T}}(x) = M_{xx}^{\mathrm{E}}(x) = -\frac{q_0 L^2}{2}\left(1 - \frac{x}{L}\right)^2, \tag{2.48}$$

$$\phi_x^{\mathrm{T}}(x) = -\frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x} = -\frac{q_0 L^3}{6 D_{xx}}\left(3\frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3}\right), \tag{2.49}$$

$$w_0^{\mathrm{T}}(x) = w_0^{\mathrm{E}} + \frac{1}{K_s A_{xz}}\left[M_{xx}^{\mathrm{E}}(x) - M_{xx}^{\mathrm{E}}(0)\right] = \frac{q_0 L^4}{24 D_{xx}}\left[\left(6\frac{x^2}{L^2} - 4\frac{x^3}{L^3} + \frac{x^4}{L^4}\right) + 12\Omega\frac{x}{L}\left(2 - \frac{x}{L}\right)\right]. \tag{2.50}$$

In the LBT, the boundary conditions lead to

$$C_1 = C_2 = 0, \quad C_3 = -\frac{\alpha F_{xx}}{S_{xz}} Q_x^{\mathrm{E}}(0), \quad C_4 = \frac{\bar{D}_{xx}}{S_{xz}} M_{xx}^{\mathrm{E}}(0), \tag{2.51}$$

and the solution becomes

$$Q_x^{\mathrm{L}}(x) = Q_x^{\mathrm{E}}(x) = q_0 L\left(1 - \frac{x}{L}\right), \tag{2.52}$$

$$M_{xx}^{\mathrm{L}}(x) = M_{xx}^{\mathrm{E}}(x) = -\frac{q_0 L^2}{2}\left(1 - \frac{x}{L}\right)^2, \tag{2.53}$$

$$\phi_x^{\mathrm{L}}(x) = -\frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x} + \frac{\alpha F_{xx}}{D_{xx}S_{xz}}\left[Q_x^{\mathrm{E}}(x) - Q_x^{\mathrm{E}}(0)\right] = -\frac{q_0 L^3}{6D_{xx}}\left[\left(3\frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3}\right) + 6\alpha\Lambda\frac{x}{L}\right], \tag{2.54}$$

$$\begin{aligned} w_0^{\mathrm{L}}(x) &= w_0^{\mathrm{E}} + \frac{\bar{D}_{xx}}{D_{xx}S_{xz}}\left[M_{xx}^{\mathrm{E}}(x) - M_{xx}^{\mathrm{E}}(0)\right] + \frac{\alpha F_{xx}}{D_{xx}S_{xz}}Q_x^{\mathrm{E}}(0)x \\ &= \frac{q_0 L^4}{24D_{xx}}\left[\left(6\frac{x^2}{L^2} - 4\frac{x^3}{L^3} + \frac{x^4}{L^4}\right) + 12\bar{\Omega}\left(2\frac{x}{L} - \frac{x^2}{L^2}\right) + 24\alpha\Lambda\frac{x}{L}\right]. \end{aligned} \tag{2.55}$$

In the case of the Bickford–Reddy beam, we find the solution as

$$Q_x^{\mathrm{B}}(x) = \left(\frac{q_0\mu}{\lambda^3\cosh\lambda L}\right)[\sinh\lambda x - \lambda L\cosh\lambda(L-x)] + \frac{q_0\mu}{\lambda^2}(L-x), \tag{2.56}$$

$$M_{xx}^{\mathrm{B}}(x) = M_{xx}^{\mathrm{E}}(x) = -\frac{q_0 L^2}{2}\left(1 - \frac{x}{L}\right)^2, \tag{2.57}$$

$$\phi_x^{\mathrm{B}}(x) = -\frac{\mathrm{d}w_0^{\mathrm{E}}}{dx} + \frac{\alpha F_{xx}}{D_{xx}S_{xz}}Q_x^{\mathrm{B}}, \tag{2.58}$$

$$\begin{aligned} w_0^{\mathrm{B}}(x) = w_0^{\mathrm{E}}(x) &+ \left(\frac{q_0\mu}{2\lambda^2}\right)\left(\frac{\hat{D}_{xx}}{\hat{A}_{xz}D_{xx}}\right)(2Lx - x^2) + \left(\frac{q_0\mu}{\lambda^4\cosh\lambda L}\right)\left(\frac{\hat{D}_{xx}}{\hat{A}_{xz}D_{xx}}\right)[\cosh\lambda x \\ &+ \lambda L\sinh\lambda(L-x)] - \left(\frac{q_0\mu}{\lambda^4}\right)\left(\frac{\hat{D}_{xx}}{\hat{A}_{xz}D_{xx}}\right)\left(\frac{1 + \lambda L\sinh\lambda L}{\cosh\lambda L}\right). \end{aligned} \tag{2.59}$$

3. Plates

3.1. Kirchhoff plate theory

The governing equation of equilibrium of the Kirchhoff plate theory is given by

$$\frac{\partial^2 M_{xx}^{\mathrm{K}}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{\mathrm{K}}}{\partial x\,\partial y} + \frac{\partial^2 M_{yy}^{\mathrm{K}}}{\partial y^2} + q = 0, \tag{3.1}$$

where q is the transverse load, and superscript K refers to quantities in the Kirchhoff plate theory, and

$$\begin{aligned} M_{xx}^{\mathrm{K}} &= -D\left(\frac{\partial^2 w_0^{\mathrm{K}}}{\partial x^2} + v\frac{\partial^2 w_0^{\mathrm{K}}}{\partial y^2}\right), \\ M_{yy}^{\mathrm{K}} &= -D\left(v\frac{\partial^2 w_0^{\mathrm{K}}}{\partial x^2} + \frac{\partial^2 w_0^{\mathrm{K}}}{\partial y^2}\right), \\ M_{xy}^{\mathrm{K}} &= -D(1-v)\frac{\partial^2 w_0^{\mathrm{K}}}{\partial x\partial y}, \end{aligned} \tag{3.2}$$

where D is the flexural rigidity,

$$D = \frac{Eh^3}{12(1 - \nu^2)}. \tag{3.3}$$

In terms of the deflection w_0^{K}, Eq. (3.1) takes the form

$$D\nabla^2\nabla^2 w_0^{\mathrm{K}} = q. \tag{3.4}$$

The boundary conditions involve specifying

$$w_0^{\mathrm{K}} \quad \text{or} \quad V_n^{\mathrm{K}}, \tag{3.5}$$

$$\frac{\partial w_0^{\mathrm{K}}}{\partial n} \quad \text{or} \quad M_{nn}^{\mathrm{K}}, \tag{3.6}$$

where

$$V_n^{\mathrm{K}} = Q_n^{\mathrm{K}} + \frac{\partial M_{ns}^{\mathrm{K}}}{\partial s}, \tag{3.7a}$$

$$M_{nn}^{\mathrm{K}} = M_{xx}^{\mathrm{K}} \cos^2\theta - 2M_{xy}^{\mathrm{K}} \sin\theta\cos\theta + M_{yy}^{\mathrm{K}} \sin^2\theta, \tag{3.7b}$$

$$M_{ns}^{\mathrm{K}} = -\left(M_{xx}^{\mathrm{K}} - M_{yy}^{\mathrm{K}}\right)\sin\theta\cos\theta + M_{xy}^{\mathrm{K}}\left(\cos^2\theta - \sin^2\theta\right), \tag{3.7c}$$

$$Q_n^{\mathrm{K}} = \left(\frac{\partial M_{xx}^{\mathrm{K}}}{\partial x} + \frac{\partial M_{xy}^{\mathrm{K}}}{\partial y}\right)\cos\theta + \left(\frac{\partial M_{xy}^{\mathrm{K}}}{\partial x} + \frac{\partial M_{yy}^{\mathrm{K}}}{\partial y}\right)\sin\theta, \tag{3.7d}$$

and θ is the orientation of the unit outward normal to the boundary (measured counterclockwise).

3.2. Levinson plate theory

The displacement field in Eq. (1.1a)–(1.1c) results in the following linear strains:

$$\varepsilon_{xx}^{\mathrm{L}} = z\frac{\partial\phi_x}{\partial x} - \alpha z^3\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial^2 w_0^{\mathrm{L}}}{\partial x^2}\right), \tag{3.8a}$$

$$\varepsilon_{yy}^{\mathrm{L}} = z\frac{\partial\phi_y}{\partial y} - \alpha z^3\left(\frac{\partial\phi_y}{\partial y} + \frac{\partial^2 w_0^{\mathrm{L}}}{\partial y^2}\right), \tag{3.8b}$$

$$2\varepsilon_{xy}^{\mathrm{L}} = z\left(\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x}\right) - \alpha z^3\left(\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x} + 2\frac{\partial^2 w_0^{\mathrm{L}}}{\partial x\partial y}\right), \tag{3.8c}$$

$$\gamma_{xz}^{\mathrm{L}} = \left(1 - \beta z^2\right)\left(\phi_x + \frac{\partial w_0^{\mathrm{L}}}{\partial x}\right), \tag{3.8d}$$

$$\gamma_{yz}^{\mathrm{L}} = \left(1 - \beta z^2\right)\left(\phi_y + \frac{\partial w_0^{\mathrm{L}}}{\partial y}\right). \tag{3.8e}$$

Recall from Eq. (2.6) that $\alpha = 4/(3h^2)$ and $\beta = 4/h^2$.

The equations of equilibrium of the LPT are obtained by integrating the stress equations of equilibrium of the three-dimensional elasticity. We have

$$-\left(\frac{\partial M_{xx}^{\mathrm{L}}}{\partial x}+\frac{\partial M_{xy}^{\mathrm{L}}}{\partial y}\right)+Q_x^{\mathrm{L}}=0, \tag{3.9}$$

$$-\left(\frac{\partial M_{xy}^{\mathrm{L}}}{\partial x}+\frac{\partial M_{yy}^{\mathrm{L}}}{\partial y}\right)+Q_y^{\mathrm{L}}=0, \tag{3.10}$$

$$-\left(\frac{\partial Q_x^{\mathrm{L}}}{\partial x}+\frac{\partial Q_y^{\mathrm{L}}}{\partial y}\right)=q. \tag{3.11}$$

The three equations can be combined into one

$$\frac{\partial^2 M_{xx}^{\mathrm{L}}}{\partial x^2}+2\frac{\partial^2 M_{xy}^{\mathrm{L}}}{\partial x\,\partial y}+\frac{\partial^2 M_{yy}^{\mathrm{L}}}{\partial y^2}+q=0. \tag{3.12}$$

The boundary conditions involve specifying

$$w_0^{\mathrm{L}} \quad \text{or} \quad Q_n^{\mathrm{L}}, \tag{3.13}$$

$$\phi_n \quad \text{or} \quad M_{nn}^{\mathrm{L}}, \tag{3.14}$$

$$\phi_{ns} \quad \text{or} \quad M_{ns}^{\mathrm{L}}, \tag{3.15}$$

where $(M_{nn}^{\mathrm{L}}, M_{ns}^{\mathrm{L}}, Q_n^{\mathrm{L}})$ are defined by Eqs. (3.7b)–(3.7d) with superscript K replaced with L.

The moment resultants $(M_{xx}^{\mathrm{L}}, M_{yy}^{\mathrm{L}}, M_{xy}^{\mathrm{L}})$ and transverse shear stress resultants $(Q_x^{\mathrm{L}}, Q_y^{\mathrm{L}})$ are given in terms of the displacements as

$$M_{xx}^{\mathrm{L}}=\frac{D}{5}\left[4\left(\frac{\partial\phi_x}{\partial x}+v\frac{\partial\phi_y}{\partial y}\right)-\left(\frac{\partial^2 w_0^{\mathrm{L}}}{\partial x^2}+v\frac{\partial^2 w_0^{\mathrm{L}}}{\partial y^2}\right)\right], \tag{3.16a}$$

$$M_{yy}^{\mathrm{L}}=\frac{D}{5}\left[4\left(v\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right)-\left(v\frac{\partial^2 w_0^{\mathrm{L}}}{\partial x^2}+\frac{\partial^2 w_0^{\mathrm{L}}}{\partial y^2}\right)\right], \tag{3.16b}$$

$$M_{xy}^{\mathrm{L}}=\frac{D(1-v)}{5}\left[2\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)-\frac{\partial^2 w_0^{\mathrm{L}}}{\partial x\,\partial y}\right], \tag{3.16c}$$

$$Q_x^{\mathrm{L}}=\frac{2Gh}{3}\left(\phi_x+\frac{\partial w_0^{\mathrm{L}}}{\partial x}\right)=\frac{Eh}{3(1+v)}\left(\phi_x+\frac{\partial w_0^{\mathrm{L}}}{\partial x}\right), \tag{3.17a}$$

$$Q_y^{\mathrm{L}}=\frac{2Gh}{3}\left(\phi_y+\frac{\partial w_0^{\mathrm{L}}}{\partial y}\right)=\frac{Eh}{3(1+v)}\left(\phi_y+\frac{\partial w_0^{\mathrm{L}}}{\partial y}\right). \tag{3.17b}$$

3.3. Relationships

The governing equations of static equilibrium of plates according to the Kirchhoff and Levinson plate theories can be expressed in terms of the moment sum (or Marcus moment (Marcus, 1932), $\mathscr{M}$:

$$\mathscr{M}=\frac{M_{xx}+M_{yy}}{1+v}. \tag{3.18}$$

First note that

$$\mathscr{M}^{\mathrm{K}} = -D\left(\frac{\partial^2 w_0^{\mathrm{K}}}{\partial x^2} + \frac{\partial^2 w_0^{\mathrm{K}}}{\partial y^2}\right) = -D\nabla^2 w_0^{\mathrm{K}}, \tag{3.19}$$

$$\mathscr{M}^{\mathrm{L}} = \frac{D}{5}\left[4\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) - \nabla^2 w_0^{\mathrm{L}}\right]. \tag{3.20}$$

Hence, the equilibrium equation (3.6) of the Kirchhoff plate theory can be expressed as

$$\nabla^2 \mathscr{M}^{\mathrm{K}} = -q, \tag{3.21a}$$

$$\nabla^2 w_0^{\mathrm{K}} = -\frac{\mathscr{M}^{\mathrm{K}}}{D}. \tag{3.21b}$$

Similarly, the equilibrium equation (3.12) can be written as

$$\nabla^2 \mathscr{M}^{\mathrm{L}} = -q, \tag{3.22a}$$

$$\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} = \frac{1}{4}\left(\nabla^2 w_0^{\mathrm{L}} + \frac{5\mathscr{M}^{\mathrm{L}}}{D}\right). \tag{3.22b}$$

From Eqs. (3.21a) and (3.22a), in view of the load equivalence, it follows that

$$\mathscr{M}^{\mathrm{L}} = \mathscr{M}^{\mathrm{K}} + D\nabla^2 \Phi, \tag{3.23}$$

where Φ is a function such that it satisfies the biharmonic equation

$$\nabla^4 \Phi = 0. \tag{3.24}$$

Substitution of Eqs. (3.17a) and (3.17b) into Eq. (3.11) yields

$$-\frac{2}{3}Gh\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \nabla^2 w_0^{\mathrm{L}}\right) = q. \tag{3.25}$$

From Eqs. (3.22b) and (3.25), we obtain

$$\frac{5}{6Gh}\left(\nabla^2 w_0^{\mathrm{L}} + \frac{\mathscr{M}^{\mathrm{L}}}{D}\right) = -q, \tag{3.26}$$

and using Eqs. (3.19), (3.21a) and (3.23), we arrive at the expression

$$w_0^{\mathrm{L}} = w_0^{\mathrm{K}} + \frac{\mathscr{M}^{\mathrm{K}}}{\frac{5}{6}Gh} - \Phi + \Psi, \tag{3.27}$$

where Ψ is a function such that

$$\nabla^2 \Psi = 0. \tag{3.28}$$

After a series of algebraic manipulations, we can establish the following relationships for the rotations and stress resultants:

$$\phi_x = -\frac{\partial w_0^{\mathrm{K}}}{\partial x} + \frac{3}{10Gh}\frac{\partial \mathscr{M}^{\mathrm{K}}}{\partial x} + \frac{\partial \Theta}{\partial x} + \frac{h^2}{10}\frac{\partial \Omega}{\partial y}, \tag{3.29a}$$

$$\phi_y = -\frac{\partial w_0^{\mathrm{K}}}{\partial y} + \frac{3}{10Gh}\frac{\partial \mathscr{M}^{\mathrm{K}}}{\partial y} + \frac{\partial \Theta}{\partial y} - \frac{h^2}{10}\frac{\partial \Omega}{\partial x}, \tag{3.29b}$$

$$M_{xx}^{\mathrm{L}} = M_{xx}^{\mathrm{K}} - D(1-\nu)\frac{\partial}{\partial y}\left(\frac{\partial \Lambda}{\partial y} - \frac{2h^2}{25}\frac{\partial \Omega}{\partial x}\right) + D\nabla^2 \Phi, \tag{3.30a}$$

$$M_{yy}^{\mathrm{L}} = M_{yy}^{\mathrm{K}} - D(1-\nu)\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial x} + \frac{2h^2}{25}\frac{\partial \Omega}{\partial y}\right) + D\nabla^2 \Phi, \tag{3.30b}$$

$$M_{xy}^{\mathrm{L}} = M_{xy}^{\mathrm{K}} + D(1-\nu)\left[\frac{\partial^2 \Lambda}{\partial x \partial y} - \frac{h^2}{25}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\Omega\right], \tag{3.30c}$$

$$Q_x^{\mathrm{L}} = Q_x^{\mathrm{K}} + D\frac{\partial}{\partial x}\nabla^2 \Phi + \frac{2}{5}D(1-\nu)\frac{\partial \Omega}{\partial y}, \tag{3.31a}$$

$$Q_y^{\mathrm{L}} = Q_y^{\mathrm{K}} + D\frac{\partial}{\partial y}\nabla^2 \Phi - \frac{2}{5}D(1-\nu)\frac{\partial \Omega}{\partial x}, \tag{3.31b}$$

where

$$\Theta = \frac{3D}{2Gh}\nabla^2 \Phi + \Phi - \Psi, \qquad \Lambda = \frac{D}{\frac{5}{6}Gh}\nabla^2 \Phi + \Phi - \Psi, \tag{3.32}$$

and Ω is defined by

$$\Omega = \frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}, \tag{3.33a}$$

and it is the solution of

$$-\nabla^2 \Omega + \frac{10}{h^2}\Omega = 0. \tag{3.33b}$$

3.4. Lévy solutions using relationships

Here, we consider the bending problem of rectangular plates with two opposite edges simply supported while the other two edges are supported in an arbitrary manner. The coordinate system is chosen such that sides $x = 0, a$ are simply supported, while sides $y = \pm b/2$ have arbitrary boundary conditions. The solution of such plates can be obtained using Lévy's method of analysis in which the solution and load distribution are expressed in the form

$$w_0^{\mathrm{L}}(x,y) = \sum_{m=1}^{\infty} W_m^{\mathrm{L}}(y)\sin \alpha_m x, \tag{3.34}$$

$$\phi_x(x,y) = \sum_{m=1}^{\infty} X_m(y)\cos \alpha_m x, \tag{3.35}$$

$$\phi_y(x,y) = \sum_{m=1}^{\infty} Y_m(y)\sin \alpha_m x, \tag{3.36}$$

$$q(x,y) = \sum_{m=1}^{\infty} Q_m(y)\sin \alpha_m x, \tag{3.37}$$

where the edges $x = 0$, a are simply supported, $\alpha_m = (m\pi/a)$, the coefficients Q_m are determined from

$$Q_m(y) = \frac{2}{a}\int_0^a q(x,y)\sin\alpha_m x\,\mathrm{d}x \tag{3.38}$$

and a and b are the plate dimensions along the x and y axes, respectively. Clearly, the assumed solution satisfies the boundary conditions of the simply supported edges. The Lévy solution of the Kirchhoff plates is represented by

$$w_0^{\mathrm{K}}(x,y) = \sum_{m=1}^{\infty} W_m^{\mathrm{K}}(y)\sin\alpha_m x. \tag{3.39}$$

We wish to determine the unknown coefficients $(W_m^{\mathrm{L}}, X_m, Y_m)$, and consequently $(w_0^{\mathrm{L}}, \phi_x, \phi_y, Q_x^{\mathrm{L}}, Q_y^{\mathrm{L}}, M_{xx}^{\mathrm{L}}, M_{yy}^{\mathrm{L}}, M_{xy}^{\mathrm{L}})$, in terms of W_m^{K}.

Substituting the expressions (3.34)–(3.37) and (3.39) into Eqs. (3.19) and (3.20) and the results into Eq. (3.23), we arrive at

$$\mathscr{M}^{\mathrm{L}} = \mathscr{M}^{\mathrm{K}} + D\sum_{m=1}^{\infty}(C_{1m}\sinh\alpha_m y + C_{2m}\cosh\alpha_m y)\sin\alpha_m x, \tag{3.40}$$

where C_{1m} and C_{2m} are constants to be determined using the boundary conditions of edges $y = \pm b/2$. Simlarly, from Eqs. (3.27), (3.29a), (3.29b), (3.30a)–(3.30c), (3.31a) and (3.31b), we have

$$w_0^{\mathrm{L}} = w_0^{\mathrm{K}}(x,y) + \frac{\mathscr{M}^{\mathrm{K}}}{\frac{5}{6}Gh} + \sum_{m=1}^{\infty}\left[\left(C_{3m} - \frac{y}{2\alpha_m}C_{1m}\right)\cosh\alpha_m y + \left(C_{4m} - \frac{y}{2\alpha_m}C_{2m}\right)\sinh\alpha_m y\right]\sin\alpha_m x, \tag{3.41}$$

$$\begin{aligned}\phi_x = {} & -\frac{\partial w_0^{\mathrm{K}}}{\partial x} + \frac{3}{10Gh}\frac{\partial\mathscr{M}^{\mathrm{K}}}{\partial x} + \sum_{m=1}^{\infty}\left\{\alpha_m\left[\left(\frac{3D}{2Gh}C_{1m} - C_{4m}\right)\sinh\alpha_m y + \left(\frac{3D}{2Gh}C_{2m} - C_{3m}\right)\cosh\alpha_m y\right]\right.\\ & \left. + \frac{y}{2}(C_{2m}\sinh\alpha_m y + C_{1m}\cosh\alpha_m y) + \frac{h^2}{10}\lambda_m(C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\right\}\cos\alpha_m x,\end{aligned} \tag{3.42}$$

$$\begin{aligned}\phi_y = {} & -\frac{\partial w_0^{\mathrm{K}}}{\partial y} + \frac{3}{10Gh}\frac{\partial\mathscr{M}^{\mathrm{K}}}{\partial y} + \sum_{m=1}^{\infty}\left\{\alpha_m\left[\left(\frac{3D}{2Gh}C_{2m} - C_{3m}\right)\sinh\alpha_m y + \left(\frac{3D}{2Gh}C_{1m} - C_{4m}\right)\cosh\alpha_m y\right]\right.\\ & + \frac{1}{2}\left[\left(yC_{1m} + \frac{1}{\alpha_m}C_{2m}\right)\sinh\alpha_m y + \left(yC_{2m} + \frac{1}{\alpha_m}C_{1m}\right)\cosh\alpha_m y\right]\\ & \left. + \frac{h^2}{10}\alpha_m(C_{5m}\sinh\lambda_m y + C_{6m}\cosh\lambda_m y)\right\}\sin\alpha_m x,\end{aligned} \tag{3.43}$$

$$\begin{aligned}M_{xx}^{\mathrm{L}} = {} & M_{xx}^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left\{\nu(C_{1m}\sinh\alpha_m y + C_{2m}\cosh\alpha_m y) - (1-\nu)\alpha_m\left[\left(\frac{D}{\frac{5}{6}Gh}\alpha_m C_{1m} - \alpha_m C_{4m} + \frac{y}{2}C_{2m}\right)\right.\right.\\ & \left.\times\sinh\alpha_m y + \left(\frac{D}{\frac{5}{6}Gh}\alpha_m C_{2m} - \alpha_m C_{3m} + \frac{y}{2}C_{1m}\right)\cosh\alpha_m y\right] - \frac{2h^2}{25}(1-\nu)\lambda_m\alpha_m\\ & \left.\times(C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\right\}\sin\alpha_m x,\end{aligned} \tag{3.44}$$

$$M_{yy}^{\mathrm{L}} = M_{yy}^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left\{(C_{1m}\sinh\alpha_m y + C_{2m}\cosh\alpha_m y) + (1-v)\alpha_m\left[\left(\frac{D}{\frac{5}{6}Gh}\alpha_m C_{1m} - \alpha_m C_{4m} + \frac{y}{2}C_{2m}\right)\right.\right.$$
$$\left.\times \sinh\alpha_m y + \left(\frac{D}{\frac{5}{6}Gh}\alpha_m C_{2m} - \alpha_m C_{3m} + \frac{y}{2}C_{1m}\right)\cosh\alpha_m y\right] + \frac{2h^2}{25}(1-v)\lambda_m\alpha_m$$
$$\left.\times (C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\right\}\sin\alpha_m x, \tag{3.45}$$

$$M_{xy}^{\mathrm{L}} = M_{xy}^{\mathrm{K}} + D(1-v)\sum_{m=1}^{\infty}\left\{\alpha_m\left[\left(\frac{D}{\frac{5}{6}Gh}\alpha_m C_{1m} - \alpha_m C_{4m} + \frac{y}{2}C_{2m} + \frac{1}{2\alpha_m}C_{1m}\right)\cosh\alpha_m y\right.\right.$$
$$\left. + \left(\frac{D}{\frac{5}{6}Gh}\alpha_m C_{2m} - \alpha_m C_{3m} + \frac{y}{2}C_{1m} + \frac{1}{2\alpha_m}C_{2m}\right)\sinh\alpha_m y\right] + \frac{1}{25}h^2\left(\lambda_m^2 + \alpha_m^2\right)(C_{5m}\sinh\lambda_m y$$
$$\left. + C_{6m}\cosh\lambda_m y)\right\}\cos\alpha_m x, \tag{3.46}$$

$$Q_x^{\mathrm{L}} = Q_x^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left[\alpha_m(C_{1m}\sinh\alpha_m y + C_{2m}\cosh\alpha_m y) + \frac{2}{5}(1-v)\lambda_m(C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\right]$$
$$\times \cos\alpha_m x, \tag{3.47}$$

$$Q_y^{\mathrm{L}} = Q_y^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left[\alpha_m(C_{1m}\cosh\alpha_m y + C_{2m}\sinh\alpha_m y) + \frac{2}{5}(1-v)\alpha_m(C_{5m}\sinh\lambda_m y + C_{6m}\cosh\lambda_m y)\right]$$
$$\times \sin\alpha_m x, \tag{3.48}$$

where

$$\lambda_m^2 = \alpha_m^2 + \frac{10}{h^2}. \tag{3.49}$$

A total of six constants, C_{1m} through C_{6m}, are to be determined using the boundary conditions on edges $y = \pm(b/2)$ of a specific plate problem.

3.5. Examples

3.5.1. General solution

Here, we determine the six constants for the case of plates with edges $x = 0$, a simply supported and the remaining two edges, $y = \pm(b/2)$, having the same boundary conditions, namely simply supported, clamped or free. For this case, the deflection should be symmetric about the y axis when the load is symmetric in y. Therefore, only the even functions of y should be included in the expression for w_0^{L}. Thus, we must have

$$C_{1m} = C_{4m} = 0. \tag{3.50a}$$

Similarly, ϕ_y must be an odd function of y, implying

$$C_{6m} = 0. \tag{3.50b}$$

Consequently, Eqs. (3.41)–(3.48) reduce to

$$w_0^{\mathrm{L}} = w_0^{\mathrm{K}}(x,y) + \frac{\mathscr{M}^{\mathrm{K}}}{\frac{5}{6}Gh} + \sum_{m=1}^{\infty}\left(C_{3m}\cosh\alpha_m y - \frac{y}{2\alpha_m}C_{2m}\sinh\alpha_m y\right)\sin\alpha_m x, \tag{3.51}$$

$$\phi_x = -\frac{\partial w_0^{\mathrm{K}}}{\partial x} + \frac{3}{10Gh}\frac{\partial\mathscr{M}^{\mathrm{K}}}{\partial x} + \sum_{m=1}^{\infty}\left[\alpha_m\left(\frac{3D}{2Gh}C_{2m} - C_{3m}\right)\cosh\alpha_m y + \frac{y}{2}C_{2m}\sinh\alpha_m y + \frac{h^2}{10}\lambda_m C_{5m}\cosh\lambda_m y\right]\cos\alpha_m x, \tag{3.52}$$

$$\phi_y = -\frac{\partial w_0^{\mathrm{K}}}{\partial y} + \frac{3}{10Gh}\frac{\partial\mathscr{M}^{\mathrm{K}}}{\partial y} + \sum_{m=1}^{\infty}\left[\alpha_m\left(\frac{3D}{2Gh}C_{2m} - C_{3m}\right)\sinh\alpha_m y + \frac{C_{2m}}{2}\left(\frac{1}{\alpha_m}\sinh\alpha_m y + y\cosh\alpha_m y\right) + \frac{h^2}{10}\alpha_m C_{5m}\sinh\lambda_m y\right]\sin\alpha_m x, \tag{3.53}$$

$$M_{xx}^{\mathrm{L}} = M_{xx}^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left\{vC_{2m}\cosh\alpha_m y - \frac{2h^2}{25}(1-v)\lambda_m\alpha_m C_{5m}\cosh\lambda_m y - (1-v)\alpha_m\left[\frac{y}{2}C_{2m}\sinh\alpha_m y + \alpha_m\left(\frac{D}{\frac{5}{6}Gh}C_{2m} - C_{3m}\right)\cosh\alpha_m y\right]\right\}\sin\alpha_m x, \tag{3.54}$$

$$M_{yy}^{\mathrm{L}} = M_{yy}^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left\{C_{2m}\cosh\alpha_m y + \frac{2h^2}{25}(1-v)\lambda_m\alpha_m C_{5m}\cosh\lambda_m y + (1-v)\alpha_m\left[\frac{y}{2}C_{2m}\sinh\alpha_m y + \alpha_m\left(\frac{D}{\frac{5}{6}Gh}C_{2m} - C_{3m}\right)\cosh\alpha_m y\right]\right\}\sin\alpha_m x, \tag{3.55}$$

$$M_{xy}^{\mathrm{L}} = M_{xy}^{\mathrm{K}} + D(1-v)\sum_{m=1}^{\infty}\left\{\left[\frac{1}{2}\alpha_m C_{2m}\ y\cosh\alpha_m y + \left(\frac{D}{\frac{5}{6}Gh}\alpha_m^2 C_{2m} - \alpha_m^2 C_{3m} + \frac{1}{2}C_{2m}\right)\sinh\alpha_m y\right] + \frac{1}{25}h^2(\lambda_m^2 + \alpha_m^2)C_{5m}\sinh\lambda_m y\right\}\cos\alpha_m x, \tag{3.56}$$

$$Q_x^{\mathrm{L}} = Q_x^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left[\alpha_m C_{2m}\cosh\alpha_m y + \frac{2}{5}(1-v)\lambda_m C_{5m}\cosh\lambda_m y\right]\cos\alpha_m x, \tag{3.57}$$

$$Q_y^{\mathrm{L}} = Q_y^{\mathrm{K}} + D\sum_{m=1}^{\infty}\left[\alpha_m C_{2m}\sinh\alpha_m y + \frac{2}{5}(1-v)\alpha_m C_{5m}\sinh\lambda_m y\right]\sin\alpha_m x. \tag{3.58}$$

The three constants C_{2m}, C_{3m}, and C_{5m} can be determined using the actual boundary conditions on edges $y = \pm b/2$. In the paragraphs below, we shall determine them for simply supported, clamped, and free edges.

3.5.2. Simple supports on edges $y = \pm b/2$

When edges $y = \pm b/2$ of the rectangular plate are simply supported, we require

$$w_0^{\mathrm{L}} = w_0^{\mathrm{K}} = 0, \quad M_{yy}^{\mathrm{L}} = M_{yy}^{\mathrm{K}} = 0, \quad \phi_x = 0. \tag{3.59}$$

Hence from Eqs. (3.51)–(3.53) and (3.59), the constants are found to be

$$C_{2m} = C_{3m} = C_{5m} = 0. \tag{3.60}$$

In view of Eqs. (3.50a), (3.50b), (3.58) and (3.60), the bending relationships are further simplified to

$$w_0^{\mathrm{L}}(x,y) = w_0^{\mathrm{K}}(x,y) + \frac{6}{5Gh}\mathscr{M}^{\mathrm{K}}, \tag{3.61}$$

$$\phi_x(x,y) = -\frac{\partial w_0^{\mathrm{K}}}{\partial x} + \frac{3}{10Gh}\frac{\partial \mathscr{M}^{\mathrm{K}}}{\partial x}, \tag{3.62a}$$

$$\phi_y(x,y) = -\frac{\partial w_0^{\mathrm{K}}}{\partial y} + \frac{3}{10Gh}\frac{\partial \mathscr{M}^{\mathrm{K}}}{\partial y}, \tag{3.62b}$$

$$M_{xx}^{\mathrm{L}} = M_{xx}^{\mathrm{K}}, \quad M_{yy}^{\mathrm{L}} = M_{yy}^{\mathrm{K}}, \quad M_{xy}^{\mathrm{L}} = M_{xy}^{\mathrm{K}}, \tag{3.63}$$

$$Q_x^{\mathrm{L}} = Q_x^{\mathrm{K}} \equiv \frac{\partial M_{xx}^{\mathrm{K}}}{\partial x} + \frac{\partial M_{xy}^{\mathrm{K}}}{\partial y}, \tag{3.64a}$$

$$Q_y^{\mathrm{L}} = Q_y^{\mathrm{K}} \equiv \frac{\partial M_{xy}^{\mathrm{K}}}{\partial x} + \frac{\partial M_{yy}^{\mathrm{K}}}{\partial y}. \tag{3.64b}$$

From Eqs. (3.63), (3.64a) and (3.64b), one can see that for simply supported rectangular plates, the stress resultants predicted by the Levinson plate theory are equal to those furnished by the classical thin plate theory. In addition, considering Eq. (3.33a), the solution effectively leads to a constant in-plane rotation tensor, i.e.,

$$\frac{\partial \phi_x}{\partial y} = \frac{\partial \phi_y}{\partial x}. \tag{3.65}$$

Lim et al. (1988) made the assumption that the relation given by Eq. (3.65) is true for plate problems using the Lévy method of analysis. However, it should be pointed out that the relation given by Eq. (3.65) is only valid for rectangular plates with all edges simply supported. It is not valid for rectangular plates with edges $x = 0$, a simply supported while edges $y = \pm b/2$ are either clamped or free.

Table 1 gives the maximum deflection parameters of simply supported rectangular plates subjected to a uniformly distributed load q_0. As observed from Table 1, there is an excellent agreement between the present Levinson results obtained using Eq. (3.61) and those exact solutions of Cooke and Levinson (1983) obtained from solving the uncoupled governing equations of the LPT. The agreement of results confirms

Table 1
Maximum deflection parameters ($\bar{w} - w_0 D/q_0 a^4$) for simply supported rectangular plates under uniformly distributed load q_0 ($v = 0.3$, $K_s = 5/6$ and $m = 20$)

h/a	$b/a = 1$ [a]			$b/a = 2$ [a]		
	Lee et al. (2000) [b]	Cooke and Levinson (1983) [c]	Eq. (3.61) [c]	Lee et al. (2000) [b]	Cooke and Levinson (1983) [c]	Eq. (3.61) [c]
0.04	0.00410	0.00410	0.00410	0.01018	0.01018	0.01018
0.10	0.00427	0.00427	0.00427	0.01045	0.01045	0.01045
0.20	0.00490	0.00490	0.00490	0.01143	0.01143	0.01143

[a] Thin plate solutions are 0.00406 and 0.01013 for $b/a = 1$ and 2, respectively.
[b] Based on the Mindlin plate theory.
[c] Based on the LPT.

the correctness of the bending relationships derived. Interestingly, one can see from Table 1 that if the value of 5/6 is adopted for the Mindlin shear correction factor, the Mindlin plate deflections of simply supported rectangular plates are exactly the same as those based on the LPT. One can also show that (Lee et al., 2000) the Mindlin and Levinson stress resultants will also be the same for rectangular plates if the Mindlin shear correction factor K_s is taken to be 5/6.

3.5.3. Clamped on edges $y = \pm b/2$

When edges $y = \pm b/2$ of the rectangular plate are clamped, we use the boundary conditions

$$w_0^{\mathrm{L}} = w_0^{\mathrm{K}} = 0, \quad \phi_y = \frac{\partial w_0^{\mathrm{K}}}{\partial y} = 0, \quad \phi_x = 0. \tag{3.66}$$

Substituting the boundary conditions (3.66) in Eqs. (3.51)–(3.53), the constants are found to be

$$C_{2m} = \frac{\alpha_m\left[\rho_m\lambda_m \tanh\frac{\alpha_m b}{2} - (\alpha_m\rho_m + \xi_m)\tanh\frac{\lambda_m b}{2}\right] + \lambda_m\eta_m}{\left(A_m\lambda_m \sinh\frac{\alpha_m b}{2} - B_m\cosh\frac{\alpha_m b}{2}\right)}, \tag{3.67a}$$

$$C_{3m} = \operatorname{sech}\frac{\alpha_m b}{2}\left(\frac{b}{4\alpha_m}C_{2m}\sinh\frac{\alpha_m b}{2} - \rho_m\right), \tag{3.67b}$$

$$C_{5m} = -\frac{10}{h^2}\left(\frac{1}{\lambda_m}\right)\operatorname{sech}\frac{\lambda_m b}{2}\left[\alpha_m\left(\frac{3D}{2Gh}\right)C_{2m}\cosh\frac{\alpha_m b}{2} + \alpha_m\rho_m + \xi_m\right], \tag{3.67c}$$

where

$$A_m = \left[\frac{b}{4}\tanh\frac{\alpha_m b}{2} - \alpha_m\left(\frac{3D}{2Gh}\right) - \frac{1}{2\alpha_m}\right], \qquad B_m = \left[\frac{\lambda_m b}{4} - \alpha_m^2\left(\frac{3D}{2Gh}\right)\tanh\frac{\lambda_m b}{2}\right], \tag{3.68a}$$

$$\rho_m = \left(\frac{6}{5Gh}\right)\mathscr{M}_m^{\mathrm{K}}|_{y=b/2}, \qquad \xi_m = \left(\frac{3}{10Gh}\right)Q_{xm}^{\mathrm{K}}|_{y=b/2}, \qquad \eta_m = \left(\frac{3}{10Gh}\right)Q_{ym}^{\mathrm{K}}|_{y=b/2}. \tag{3.68b}$$

Table 2 gives the maximum deflection parameters of rectangular plates with edges $x = 0, a$ simply supported and the other two edges, $y = \pm b/2$, clamped. The plate is subjected to a uniformly distributed load of intensity q_0. A disparity of the Levinson plate results can be observed between those furnished by Cooke and Levinson (1983) and by the present relationship (Eqs. (3.51), (3.67a)–(3.67c), (3.68a) and (3.68b)). Even when compared to the Mindlin plate solutions obtained by Lee et al. (2000), where the normals are assumed to undergo constant rotations, the Levinson plate results by Cooke and Levinson (1983) are consistently lower and hence their Levinson plate seems to be stiffer. This is contrary to Levinson's more flexible plate formulation as the plate theory allows the warping of the rotated normals. To

Table 2
Maximum deflection parameters ($\bar{w} = w_0 D/(q_0 a^4)$) for rectangular plates with two opposite edges and the other two edges clamped, and subjected to uniformly distributed load q_0 ($\nu = 0.3$, $K_s = 5/6$ and $m = 20$).

h/a	$b/a = 1$[a]			$b/a = 2$[a]		
	Lee et al. (2000)[b]	Cooke and Levinson (1983)[c]	Eq. (3.61)[c]	Lee et al. (2000)[b]	Cooke and Levinson (1983)[c]	Eq. (3.61)[c]
0.04	0.00197	0.00196	0.00198	0.00851	0.00850	0.00852
0.10	0.00221	0.00217	0.00227	0.00885	0.00879	0.00889
0.20	0.00302	0.00292	0.00322	0.01000	0.00984	0.01013

[a] Thin plate solutions are 0.00192 and 0.00845 for $b/a = 1$ and 2, respectively.
[b] Based on the Mindlin plate theory.
[c] Based on the LPT.

seek an explanation for the disparity of results, one may first derive the uncoupled equations for the Levinson transverse deflection and the normal rotations, which are given as

$$\nabla^4 w_0^{\mathrm{L}} = \left(\frac{1}{D} - \frac{6}{5Gh}\nabla^2\right)q, \tag{3.69a}$$

$$\nabla^4 \phi_x = -\left(\frac{1}{D} + \frac{3}{10Gh}\nabla^2\right)\frac{\partial q}{\partial x} + \frac{10}{h^2}\frac{\partial}{\partial y}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right), \tag{3.69b}$$

$$\nabla^4 \phi_y = -\left(\frac{1}{D} + \frac{3}{10Gh}\nabla^2\right)\frac{\partial q}{\partial y} - \frac{10}{h^2}\frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right). \tag{3.69c}$$

The uncoupled equation for the Levinson transverse deflection, obtained herein is the same as that derived by Cooke and Levinson (1983). However, the uncoupled equations for the normal rotations presented by Cooke and Levinson (1983) are erroneous, as their equations do not contain the last term on the right-hand side of Eqs. (3.69b) and (3.69c). As discussed earlier, this last term can only be omitted for the special case of a simply supported rectangular plate where these missing terms vanish inherently, as shown in Eq. (3.65). As one can see by ignoring these terms or by imposing the constant in-plane rotation tensor on the concerned plate problem, it will have an undesirable stiffening effect on the plate bending behavior.

3.5.4. Free on edges $y = \pm b/2$

When edges $y = \pm b/2$ of the rectangular plate are free, the boundary conditions are

$$M_{yy}^{\mathrm{L}} = M_{yy}^{\mathrm{K}} = 0, \quad Q_y^{\mathrm{L}} = V_y^{\mathrm{K}} = 0, \quad M_{xy}^{\mathrm{L}} = 0, \tag{3.70}$$

where $V_y^{\mathrm{K}} = Q_y^{\mathrm{K}} + \partial M_{xy}^{\mathrm{K}}/\partial x$ is the Kirchhoff effective shear force. In view of these boundary conditions, Eqs. (3.55), (3.56) and (3.58) give

$$C_{2m} = \frac{\rho_m \lambda_m \coth\frac{\lambda_m b}{2} + \left[\xi_m - \frac{\rho_m}{2\alpha_m}\left(\lambda_m^2 + \alpha_m^2\right)\right]\coth\frac{\alpha_m b}{2}}{\left(A_m \sinh\frac{\alpha_m b}{2} - B_m \cosh\frac{\alpha_m b}{2}\right)}, \tag{3.71a}$$

$$C_{3m} = \frac{1}{\alpha_m^2}\operatorname{cosech}\frac{\alpha_m b}{2}\left\{\left[\frac{\alpha_m b}{4}\cosh\frac{\alpha_m b}{2} - \frac{1+v}{2(1-v)}\sinh\frac{\alpha_m b}{2}\right]C_{2m} - \frac{1}{2\alpha_m}\left(\lambda_m^2 + \alpha_m^2\right)\rho_m + \xi_m\right\}, \tag{3.71b}$$

$$C_{5m} = -\frac{25}{2h^2}\operatorname{cosech}\frac{\lambda_m b}{2}\left(\frac{6D}{5Gh}C_{2m}\sinh\frac{\alpha_m b}{2} + \frac{\rho_m}{\alpha_m}\right), \tag{3.71c}$$

$$A_m = \left[\frac{b}{4} - \lambda_m\left(\frac{6D}{5Gh}\right)\coth\frac{\lambda_m b}{2}\right]\alpha_m, \tag{3.71d}$$

$$B_m = \left[\frac{\alpha_m b}{4}\coth\frac{\alpha_m b}{2} - \alpha_m^2\left(\frac{6D}{5Gh}\right) - \frac{3+v}{2(1-v)}\right], \tag{3.71e}$$

$$\rho_m = \frac{6}{5Gh}\left(Q_{ym}^{\mathrm{K}}|_{y=b/2}\right), \qquad \xi_m = \frac{M_{xym}^{\mathrm{K}}|_{y=b/2}}{D(1-v)}. \tag{3.72}$$

Interestingly, if one is to substitute Eqs. (3.71a)–(3.71e) and (3.72) into Eqs. (3.51) and (3.54)–(3.58), the specialized Kirchhoff–Levinson deflection and stress–resultant relationships for rectangular plates with two opposite simply supported edges and the other two edges free will be exactly the same as the corresponding Kirchhoff–Mindlin bending relationships obtained by Lee et al. (2000) when the Mindlin shear correction factor K_s is set to 5/6. Table 3 gives results predicted by the Mindlin plate theory (Lee et al., 2000) and the LPT (via Eqs. (3.51), (3.54)–(3.58), (3.71a)–(3.71e) and (3.72)) for uniformly loaded square plates with two simply supported edges while the other two edges are free. Here, v is taken to be 0.3 while the Mindlin shear correction factor K_s is 5/6. It is evident from Table 3 that the results predicted by the Levinson and the

Table 3
Bending parameters $\bar{w} = w_0 D \times 10/(q_0 a^4)$ and $\bar{M} = M \times 10/(q_0 a^2)$ for uniformly loaded, square plates of length a with two opposite edges simply supported and the other two edges free ($\nu = 0.3$, $K_s = 5/6$ and $m = 50$)

h/a	$\bar{w}(a/2,0)$		$\bar{w}(a/2,a/2)$		$\bar{M}_{xx}(a/2,0)$		$\bar{M}_{yy}(a/2,0)$		$Q_x(0,0)/(q_0 a)$	
	Lee et al. (2000)[a]	Present work[b]	Lee et al. (2000)	Present work	Lee et al. (2000)	Present work	Lee et al. (2000)	Present work	Lee et al. (2000)	Present work
0.01	0.1310	0.1310	0.1504	0.1504	1.225	1.225	0.270	0.270	0.464	0.464
0.05	0.1319	0.1319	0.1522	0.1522	1.225	1.225	0.264	0.265	0.463	0.463
0.10	0.1346	0.1346	0.1560	0.1560	1.225	1.225	0.256	0.256	0.461	0.461
0.15	0.1391	0.1391	0.1616	0.1616	1.226	1.226	0.247	0.247	0.459	0.459
0.20	0.1454	0.1454	0.1690	0.1690	1.229	1.229	0.237	0.237	0.458	0.458

[a] Based on the Mindlin plate theory (from the unpublished manuscript of Lee et al.).
[b] Based on the LPT (using Eqs. (3.51), (3.54), (3.55), (3.57), (3.71a), (3.71e) and (3.72)).

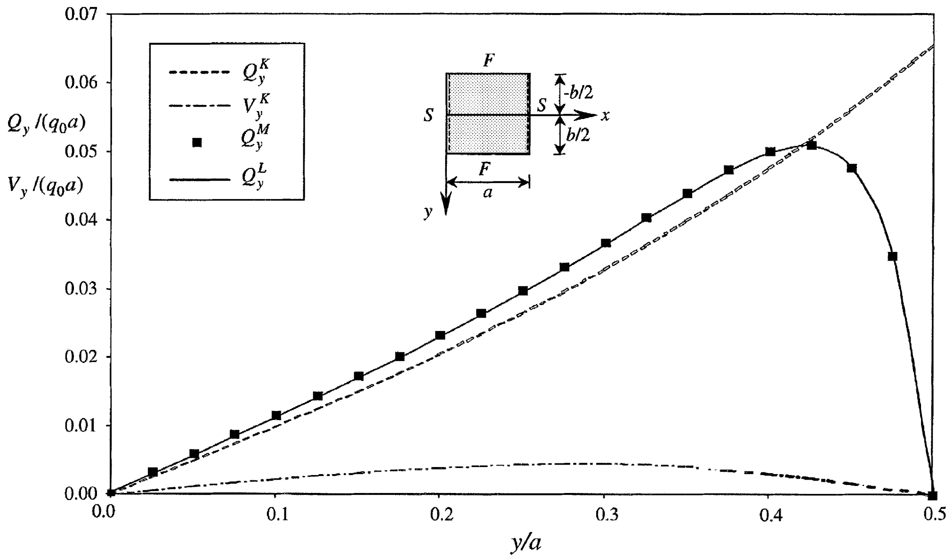

Fig. 2. Variation of shear forces along $x = a/2$ for uniformly loaded square SFSF plates ($h/a = 0.1$, $\nu = 0.3$, $K_s = 5/6$, and $m = 50$).

Mindlin plate theories have a perfect one-to-one correspondence, thus quantitatively confirming the foregoing discussion that the Kirchhoff–Levinson and the Kirchhoff–Mindlin deflection and stress–resultant relationships are the same. Also from Fig. 2, one can observe a very good agreement between the Mindlin and Levinson shear forces (Q_y^M, Q_y^L), which vanish at the free edges. The Kirchhoff shear and effective shear forces (Q_y^K, V_y^K) are also shown in Fig. 2.

4. Concluding remarks

In this paper, exact relationships between the bending solutions of the Levinson beam and plate theories and the Euler–Bernoulli beam and Kirchhoff plate theories are presented. The relationships can be used to generate bending solutions of the Levinson beam and plate theories whenever the Euler–Bernoulli beam and Kirchhoff plate solutions are available. Since solutions of the Euler–Bernoulli beam and Kirchhoff plate theories are easily determined or are available in most textbooks on mechanics of materials for a variety of boundary conditions, the relationships presented herein make it easier to compute the solutions of the Levinson beam and plate theories directly from the known classical beam and plate solutions. These relationships may also be used to develop finite element models of the Levinson beam and plate theories from those of the CPT.

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